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VARIATION THROUGH ENRICHMENT*

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Introduction

This paper continues the authors' various works [3, 4, 12, 14] on categories enriched in bicategories. We treat the elements of the theory again, here from a more algebraic (logical) and less geometric viewpoint. For a bicategory $\mathscr{W}$ we first develop $\mathscr{W}$-matrices before passing on to $\mathscr{W}$-modules, an approach which allows a simple proof of the cocompleteness of the 2-category $\mathscr{W}$-Cat of $\mathscr{W}$-categories. When $\mathscr{W}$ has precisely one object (and so is a monoidal category) the main results are in works of Bénabou [2], Lawvere [6], and Wolff [15], although a uniform treatment even in this case has not been published.

The second part of the paper relates *variable categories* with *enriched categories*. For the purposes of this paper a variable category is taken to mean a fibration over a fixed parameter category **C**. We show that the domain of variation can be organized into a bicategory $\mathscr{W}(\mathbf{C})$ such that categories varying over **C** and $\mathscr{W}(\mathbf{C})$-enriched categories appear on opposing sides of a biadjunction which tries very hard to be a biequivalence. In fact, if we impose the mild completeness condition of splitting idempotents on the fibres of the variable categories, the adjunction does restrict to a biequivalence with the "cauchy-complete" $\mathscr{W}(\mathbf{C})$-categories.

Our terminology for bicategories and 2-categories is that of [5] and [10].

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1. Matrices and graphs over a bicategory

Let Set denote the category of small sets.

A bicategory $\mathscr{W}$ is said to be *locally small-cocomplete* when each hom-category $\mathscr{W}(U,V)$ has small colimits and, for all arrows $f: U' \to U$, $g: V \to V'$ in $\mathscr{W}$, the functor $\mathscr{W}(f,g): \mathscr{W}(U,V) \to \mathscr{W}(U',V')$ preserves small colimits.

Let $\mathscr{W}$ denote a locally small-cocomplete bicategory with a small set $\mathscr{U}$ of objects.

The category Set/$\mathscr{U}$ has as objects families X of small sets X_U indexed by $U \in \mathscr{U}$; an element $x \in X_U$ is called *an element of X over U.*

The bicategory $\mathscr{W}$-Mat of *$\mathscr{W}$-matrices* is defined as follows. The objects are the objects of Set/$\mathscr{U}$. An arrow $S: X \to Y$ assigns to each pair x, y of elements of X, Y over U, V, respectively, an arrow $S(y,x): U \to V$ in $\mathscr{W}$. A 2-cell $\sigma: S \to S'$ is a family of 2-cells $\sigma_{y,x}: S(y,x) \to S'(y,x)$ in $\mathscr{W}$. Composition of 2-cells $S \to S' \to S''$ is componentwise that of $\mathscr{W}$. Composition of arrows

$$X \xrightarrow{S} Y \xrightarrow{T} Z$$

is "matrix multiplication":

$$(TS)(z,x) = \sum_{y \in Y} T(z,y)S(y,x).$$

The latter composition is compatible with 2-cells; it is associative and has identities up to coherent natural isomorphisms.

Small colimits in $\mathscr{W}$-Mat(X,Y) are constructed componentwise in the hom-categories of $\mathscr{W}$. It follows that $\mathscr{W}$-Mat is *locally small-cocomplete.*

There is a homomorphism of bicategories

$$\text{Set}/\mathscr{U} \to \mathscr{W}\text{-Mat}$$

which is the identity on objects and takes an arrow $h: X \to Y$ in Set/$\mathscr{U}$ to the matrix $h_*: X \to Y$ given as follows

$$h_*(y,x) = \begin{cases} 1_U: U \to U & \text{when } y = hx, \\ 0: U \to V & \text{otherwise,} \end{cases}$$

where x, y are elements over U, V and 0 denotes the initial object of the category $\mathscr{W}(U,V)$. Matrices of the form $h_*: X \to Y$ have right adjoints $h^*: Y \to X$ in $\mathscr{W}$-Mat: the formula for h^* is the reverse of that for h_*. (In general, not all arrows with right adjoints in $\mathscr{W}$-Mat are of the form h_*.) If h is monic then the unit $1_X \to h^*h_*$ is invertible. If h is epic the the counit $h_*h^* \to 1$ is a retraction. (The converses of the last two sentences are also true provided $\mathscr{W}$ has no objects whose identity arrows are initial.)

For each small set Y over $\mathscr{U}$ there is a category $\mathscr{P}Y$ over $\mathscr{U}$ whose objects over U are functions S which assign to each element y of Y an arrow $S(y): U \to V$ where y is over V, and whose arrows over U are families of 2-cells in $\mathscr{W}$. There is a pseudo-natural equivalence of categories:

$$\mathcal{W}\text{-Mat}(X, Y) \simeq (\mathrm{CAT}/\mathcal{V}\,)(X, \mathcal{P}Y)$$

where CAT is a suitably large 2-category of categories.

Proposition 1. *The homomorphism of bicategories*

$$\mathrm{Set}/\mathcal{V} \to \mathcal{W}\text{-Mat}$$

preserves bicolimits. The initial object 0 *of* Set/$\mathcal{V}$ *is biterminal in* $\mathcal{W}$-Mat. *For all objects* X, Y *of* Set/$\mathcal{V}$, *the coproduct diagram*

$$X \xrightarrow{\;i\;} X+Y \xleftarrow{\;j\;} Y$$

has the following properties:

(a) i^*j_*, j^*i_* *are initial in* $\mathcal{W}$-Mat(Y, X), $\mathcal{W}$-Mat(X, Y), *and the units* $1_X \to i^*i_*$, $1_Y \to j^*j_*$ *are invertible.*

(b) *The 2-cell* $i_*i^*+j_*j^* \to 1_{X+Y}$, *induced by the counits, is invertible.*

(c) *The diagram*

$$X \xleftarrow{\;i^*\;} X+Y \xrightarrow{\;j^*\;} Y$$

is a biproduct in $\mathcal{W}$-Mat.

Proof. The assignment $Y \mapsto \mathcal{P}Y$ provides a relative right biadjoint for Set/$\mathcal{V} \to$ $\mathcal{W}$-Mat modulo a change of universe. This suffices for the first sentence of the Proposition. The second sentence is trivial.

The units in (a) are invertible since i and j are monic. The remainder of (a) follows from the fact that the pushout

$$\begin{array}{ccc} 0 & \longrightarrow & X \\ \big\downarrow & & \big\downarrow{\scriptstyle i} \\ Y & \underset{j}{\longrightarrow} & X+Y \end{array}$$

becomes a bipushout in $\mathcal{W}$-Mat.

Since $X+Y$ is a bicoproduct, the 2-cell of (b) is invertible if and only if its composites with both i_* and j_* are invertible. But the composite with i_* is the composite isomorphism:

$$(i_*i^*+j_*j^*)i_* \cong i_*i^*i_*+j_*j^*i_* \cong i_*1+j_*0 \cong i_*.$$

Similarly, the composite with j_* is invertible. This gives (b).

Given $S: Z \to X$, $T: Z \to Y$ in $\mathcal{W}$-Mat, we obtain $i_*S+j_*T: Z \to X+Y$ with:

$$i^*(i_*S+j_*T) \cong i^*i_*S+i^*j_*T \cong S,$$

$$j^*(i_*S+j_*T) \cong j^*i_*S+j^*j_*T \cong T.$$

The 2-cell condition is also easily checked, yielding (c). □

A *$\mathcal{W}$-graph* $\mathcal{G}$ is a "square" matrix; that is, an endo-arrow in $\mathcal{W}$-Mat. The object of Set/$\mathcal{W}$ *and* the matrix from it to itself will both be denoted by $\mathcal{G}$. So, for each object U of $\mathcal{W}$, we have a small set $\mathcal{G}_U$ of objects of $\mathcal{G}$ over U; and, for objects A, B of $\mathcal{G}$ over U, V, we have an arrow $\mathcal{G}(B, A): U \to V$ in $\mathcal{W}$. *An arrow* $H: \mathcal{G} \to \mathcal{G}'$ *of $\mathcal{W}$-graphs* consists of an arrow $H: \mathcal{G} \to \mathcal{G}'$ in Set/$\mathcal{W}$ together with a 2-cell

$$H: H_*\mathcal{G}H^* \to \mathcal{G}'$$

in $\mathcal{W}$-Mat. So, for each object A of $\mathcal{G}$ over U, we have an object HA of $\mathcal{G}'$ over U, and, for objects A, B of $\mathcal{G}$ over U, V, we have a 2-cell

$$H_{BA}: \mathcal{G}(B, A) \to \mathcal{G}'(HB, HA)$$

in $\mathcal{W}$. This defines a category $\mathcal{W}$-Gph of $\mathcal{W}$-graphs.

Proposition 2. *The category $\mathcal{W}$-Gph has small colimits.*

Proof. Suppose $D: \mathcal{C} \to \mathcal{W}$-Gph is a functor from a small category $\mathcal{C}$. Let X denote the colimit of the composite of D with the forgetful functor $\mathcal{W}$-Gph$\to$Set/$\mathcal{W}$. There are coprojections $HC: DC \to X$ in Set/$\mathcal{W}$. There is a functor

$$\mathcal{C} \to (\mathcal{W}\text{-Mat})(X, X)$$

which takes $n: C \to C'$ to the composite

$$(HC)_*(DC)(HC)^* \xrightarrow[\cong]{} (HC')_*(Dn)_*(DC)(Dn)^*(HC')$$

$$\xrightarrow[(HC')_*(Dn)(HC')^*]{} (HC')_*(DC')(HC')^*.$$

The colimit of the last functor gives an endo-arrow of X and hence determines a $\mathcal{W}$-graph $\mathcal{G}$. The coprojections HC together with the coprojections $(HC)_*(DC)(HC)^* \to \mathcal{G}$ determine arrows $HC: DC \to \mathcal{G}$ in $\mathcal{W}$-Gph which can be checked to provide the coprojections of a colimit for D. □

2. Categories enriched over a bicategory

The following definitions occur in an equivalent, but more usual, form in [12].

A $\mathcal{W}$-category $\mathcal{A}$ is a $\mathcal{W}$-graph $\mathcal{A}$ together with 2-cells $\eta: 1 \to \mathcal{A}$, $\mu: \mathcal{A}\mathcal{A} \to \mathcal{A}$ in $\mathcal{W}$-Mat which satisfy the axioms for a monad in $\mathcal{W}$-Mat. Note that $\mathcal{A}_U$ becomes the set of objects for a category whose arrows $f: A \to B$ are 2-cells $1_U \to \mathcal{A}(A, B)$ in $\mathcal{W}$ and whose composition is determined by μ. It will be convenient to write $\mathcal{A}_U$ for this category and not merely for the set of objects of $\mathcal{A}$ over U.

A $\mathcal{W}$-functor $F: \mathcal{A} \to \mathcal{B}$ between $\mathcal{W}$-categories $\mathcal{A}, \mathcal{B}$ is an arrow of $\mathcal{W}$-graphs which respects η, μ. The arrow $F_*: \mathcal{A} \to \mathcal{B}$ and 2-cell $\bar{F}: F_*\mathcal{A} \to \mathcal{B}F_*$ (corresponding to $F: F_*\mathcal{A}F^* \to \mathcal{B}$ under $F_* \dashv F^*$) determine a "monad opfunctor" in $\mathcal{V}$-Mat (in the terminology of [9]).

For $\mathcal{V}$-functors $F, G: \mathcal{A} \to \mathcal{B}$, a ***$\mathcal{V}$-natural transformation*** $\theta: F \to G$ is a 2-cell $\theta: F_*\mathcal{A} \to \mathcal{B}G_*$ in $\mathcal{V}$-Mat such that the following diagram commutes.

$$\begin{array}{ccccc} F_*\mathcal{A}\mathcal{A} & \xrightarrow{\theta\mathcal{A}} & \mathcal{B}G_*\mathcal{A} & \xrightarrow{\mathcal{B}\tilde{G}} & \mathcal{B}\mathcal{B}G_* \\ {\scriptstyle \tilde{F}\mathcal{A}}\downarrow & & & & \downarrow{\scriptstyle \mu G_*} \\ \mathcal{B}F_*\mathcal{A} & \xrightarrow[\mathcal{B}\theta]{} & \mathcal{B}\mathcal{B}G_* & \xrightarrow[\mu G_*]{} & \mathcal{B}G_* \end{array}$$

Notice that there is a bijection between such θ and 2-cells $\bar{\theta}: F_* \to \mathcal{B}G_*$ satisfying

$$\mu G_* \cdot \mathcal{B}\bar{\theta} \cdot \tilde{F} = \mu G_* \cdot \mathcal{B}\tilde{G} \cdot \bar{\theta}\mathcal{A};$$

the bijection is given by the equations:

$$\bar{\theta} = \theta \cdot F_*\eta, \qquad \theta = \mu G_* \cdot \mathcal{B}\bar{\theta} \cdot \tilde{F}.$$

With obvious compositions, we have defined a 2-category $\mathcal{V}$-Cat of $\mathcal{V}$-categories, $\mathcal{V}$-functors and $\mathcal{V}$-natural transformations.

A monad $m: U \to U$ in the bicategory $\mathcal{V}$ can be identified with a $\mathcal{V}$-category $\mathcal{A}$ which has precisely one object A such that A is over U and $\mathcal{A}(A, A) = m$. In particular, each object U of $\mathcal{V}$ determines *a $\mathcal{V}$-category which we also denote by U* corresponding to the identity monad on U. There is an obvious isomorphism of categories:

$$(\mathcal{V}\text{-Cat})(U, \mathcal{A}) \cong \mathcal{A}_U.$$

Proposition 3. *The forgetful functor from the category* $|\mathcal{V}$-Cat$|$ *of $\mathcal{V}$-categories and $\mathcal{V}$-functors to the category* $\mathcal{V}$-Gph *has a left adjoint $\mathcal{F}$ whose value at a square matrix $\mathcal{G}: X \to X$ is the geometric series $\sum_{n \in \mathbb{N}} \mathcal{G}^n : X \to X$.*

Proof. The monoidal category $\mathcal{V}$-Mat(X, X), whose tensor-product (that is, composition) preserves small colimits, is such that the free monoid on an object $\mathcal{G}$ is $\sum \mathcal{G}^n = \mathcal{F}\mathcal{G}$. The identity of X together with the coprojection $\mathcal{G} \to \mathcal{F}\mathcal{G}$ for $n = 1$ provide an arrow $N: \mathcal{G} \to \mathcal{F}\mathcal{G}$ of $\mathcal{V}$-graphs. Suppose $H: \mathcal{G} \to \mathcal{B}$ is an arrow of $\mathcal{V}$-graphs into a $\mathcal{V}$-category $\mathcal{B}$. Then $H^*\mathcal{B}H_*: X \to X$ is a monoid in $\mathcal{V}$-Mat(X, X), so the arrow $\mathcal{G} \to H^*\mathcal{B}H_*$ (arising from H) extends to a unique monoid arrow $\mathcal{F}\mathcal{G} \to H^*\mathcal{B}H_*$ which, together with H on objects, determines a unique $\mathcal{V}$-functor $H': \mathcal{F}\mathcal{G} \to \mathcal{B}$ with $H'N = H$. □

Lemma 4. *Suppose $F, G: \mathcal{A} \to \mathcal{B}$ are monoid arrows in* $\mathcal{V}$-Mat(X, X) *and let $H: \mathcal{B} \to \mathcal{C}$ be the coequalizer of F, G in* $\mathcal{V}$-Mat(X, X). *The $\mathcal{V}$-graph $\mathcal{C}$ possesses a unique monoid structure such that H becomes a monoid arrow if and only if*

$H\cdot\mu\cdot\mathcal{B}F = H\cdot\mu\cdot\mathcal{B}G$ and $H\cdot\mu\cdot F\mathcal{B} = H\cdot\mu\cdot G\mathcal{B}$. Furthermore, in this case, this monoid arrow is a coequalizer of F, G in $|\mathcal{V}\text{-Cat}|$.

Proof. Composition in $\mathcal{V}$-Mat preserves coequalizers, so the rows and columns of the following diagram are all coequalizers.

$$\begin{array}{ccccc}
\mathcal{A}\mathcal{A} & \rightrightarrows & \mathcal{A}\mathcal{B} & \longrightarrow & \mathcal{A}\mathcal{C} \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\mathcal{B}\mathcal{A} & \rightrightarrows & \mathcal{B}\mathcal{B} & \longrightarrow & \mathcal{B}\mathcal{C} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}\mathcal{A} & \rightrightarrows & \mathcal{C}\mathcal{B} & \longrightarrow & \mathcal{C}\mathcal{C}
\end{array}$$

The existence of a unique $\mu:\mathcal{C}\mathcal{C}\to\mathcal{C}$ such that the square

$$\begin{array}{ccc}
\mathcal{B}\mathcal{B} & \xrightarrow{HH} & \mathcal{C}\mathcal{C} \\
{\scriptstyle\mu}\downarrow & & \downarrow{\scriptstyle\mu} \\
\mathcal{B} & \xrightarrow[H]{} & \mathcal{C}
\end{array}$$

commutes is equivalent to the condition that the composite

$$\mathcal{B}\mathcal{B} \xrightarrow{\mu} \mathcal{B} \xrightarrow{H} \mathcal{C}$$

should equalize both of the pairs

$$\mathcal{B}\mathcal{A} \underset{\mathcal{B}G}{\overset{\mathcal{B}F}{\rightrightarrows}} \mathcal{B}\mathcal{B}, \qquad \mathcal{A}\mathcal{B} \underset{G\mathcal{B}}{\overset{F\mathcal{B}}{\rightrightarrows}} \mathcal{B}\mathcal{B}.$$

As we must, define $\eta: 1\to\mathcal{C}$ to be $H\eta$. From the construction in Proposition 2 we see that H is the coequalizer of F, G in the category of $\mathcal{V}$-graphs. It is easy to see that $K:\mathcal{C}\to\mathcal{D}$ is a $\mathcal{V}$-functor if and only if KH is, for any arrow K of $\mathcal{V}$-graphs into a $\mathcal{V}$-category $\mathcal{D}$. □

Proposition 5. *The category* $|\mathcal{V}\text{-Cat}|$ *has coequalizers.*

Proof. Take two $\mathcal{V}$-functors $F, G:\mathcal{A}\to\mathcal{B}$ and form the coequalizer $\mathcal{C}$ of the underlying arrows of $\mathcal{V}$-graphs (Proposition 2). Let $\mathcal{D}$ be the coequalizer of the

$\mathscr{V}$-graph arrows $\mathscr{F}F, \mathscr{F}G : \mathscr{F}\mathscr{A} \to \mathscr{F}\mathscr{B}$. Then we have the following diagram in $\mathscr{V}$-Cat:

$$\begin{array}{ccccc}
\mathscr{F}\mathscr{F}\mathscr{A} & \rightrightarrows & \mathscr{F}\mathscr{F}\mathscr{B} & \longrightarrow & \mathscr{F}\mathscr{D} \\
\downarrow\downarrow & & \downarrow\downarrow & & \downarrow\downarrow \\
\mathscr{F}\mathscr{A} & \rightrightarrows & \mathscr{F}\mathscr{B} & \longrightarrow & \mathscr{F}\mathscr{C} \\
\downarrow & & \downarrow & & \\
\mathscr{A} & \underset{G}{\overset{F}{\rightrightarrows}} & \mathscr{B} & &
\end{array}$$

The category of monoids in $\mathscr{V}$-Mat(X, X) is monadic over $\mathscr{V}$-Mat(X, X) (since tensoring with a fixed object on either side preserves countable coproducts). So the first two columns of the above diagrams are coequalizers which are absolute (split) at the underlying level. Since $\mathscr{F}$ is a left adjoint, the first two rows are also coequalizers. Lemma 4 applies to the two arrows in the third column of the above diagram (since it applies to the first two columns) to yield the coequalizer of those two arrows in $|\mathscr{V}$-Cat$|$. By commutativity, an arrow from $\mathscr{B}$ into this coequalizer is induced. By the "3×3-diagram lemma" this arrow is then the coequalizer of F, G. □

Theorem 6. *The forgetful functor* $|\mathscr{V}$-Cat$| \to \mathscr{V}$-Gph *is monadic.*

Proof. Consider again the diagram in proof of Proposition 5, this time with F, G a split pair at the $\mathscr{V}$-graph level. Then the top two rows are split coequalizers. By Lemma 4 the columns are coequalizers at both the $|\mathscr{V}$-Cat$|$ *and* $\mathscr{V}$-Gph levels. By the "3×3-diagram lemma", the coequalizer of F, G is preserved by the forgetful functor. Since the forgetful functor reflects isomorphisms and in view of Proposition 3, the result follows from Beck's Theorem [8; p. 151 Ex. 6]. □

Theorem 7. *The* 2*-category* $\mathscr{V}$-Cat *admits all small colimits.*

Proof. That the category $|\mathscr{V}$-Cat$|$ has all small colimits follows from Proposition 2, Theorem 6, Proposition 5, and Linton [7; p. 81].

A monad $\mathscr{A} : X \to X$ in $\mathscr{V}$-Mat leads to a monad

$$\begin{pmatrix} \mathscr{A} & \mathscr{A} \\ 0 & \mathscr{A} \end{pmatrix} : X + X \to X + X$$

in $\mathscr{V}$-Mat which is easily verified to have the property required of $2 \otimes \mathscr{A}$ in $\mathscr{V}$-Cat;

$$(\mathscr{V}\text{-Cat})(2 \otimes \mathscr{A}, \mathscr{B}) \cong [2, \mathscr{V}\text{-Cat}(\mathscr{A}, \mathscr{B})].$$

It remains to prove that small colimits in $|\mathscr{V}$-Cat$|$ are preserved by the categoryvalued representables $\mathcal{V}$-Cat$(-,\mathcal{A})$ and hence are colimits in $\mathcal{V}$-Cat. This will follow if we can prove that the functor

$$\mathbf{2}\otimes - : |\mathcal{V}\text{-Cat}| \to |\mathcal{V}\text{-Cat}|$$

preserves small colimits. That it preserves small coproducts is trivial. That it preserves coequalizers of the type in Lemma 4 follows from the straightforward observation that the functor $\mathcal{V}$-Gph$\to\mathcal{V}$-Gph which takes

$$\mathcal{G} : X \to X \quad \text{to} \quad \begin{pmatrix} \mathcal{G} & \mathcal{G} \\ 0 & \mathcal{G} \end{pmatrix} : X+X \to X+X$$

preserves coequalizers (see Proposition 2). Using the construction of Proposition 5 and these facts, we deduce that $\mathbf{2}\otimes -$ preserves all coequalizers. □

3. Modules

Suppose $\mathcal{A}$, $\mathcal{B}$ are $\mathcal{V}$-categories; that is, monads $\mathcal{A} : X \to X$, $\mathcal{B} : Y \to Y$ in $\mathcal{V}$-Mat. Composition with $\mathcal{A}$, $\mathcal{B}$ on the right, left (respectively) determines a monad $\mathcal{V}$-Mat$(\mathcal{A},\mathcal{B})$ on the category $\mathcal{V}$-Mat(X, Y). The category of Eilenberg–Moore algebras for this monad is denoted by:

$$\mathcal{V}\text{-Mod}(\mathcal{A},\mathcal{B}).$$

An object Φ of $\mathcal{V}$-Mod$(\mathcal{A},\mathcal{B})$ is called a *$\mathcal{V}$-module* from $\mathcal{A}$ to $\mathcal{B}$; it consists of a matrix $\Phi : X \to Y$ together with compatible actions $\varrho : \Phi\mathcal{A} \to \Phi$, $\lambda : \mathcal{B}\Phi \to \Phi$.

For $\mathcal{V}$-modules $\Phi : \mathcal{A} \to \mathcal{B}$, $\Psi : \mathcal{B} \to \mathcal{C}$, there is a *composite $\mathcal{V}$-module* $\Psi\Phi : \mathcal{A} \to \mathcal{C}$ defined in the familiar "tensor-product-like" manner; that is, it is made up of the coequalizer in $\mathcal{V}$-Mat(X, Z) of the pair

$$\Psi\lambda, \varrho\Phi : \Psi\mathcal{B}\Phi \to \Psi\Phi,$$

the ϱ induced by the ϱ of Φ, and the λ induced by the λ of Ψ.

This defines a bicategory $\mathcal{V}$-Mod whose objects are $\mathcal{V}$-categories and whose arrows are $\mathcal{V}$-modules.

The category $\mathcal{V}$-Mod$(\mathcal{A},\mathcal{B})$ has small colimits since $\mathcal{V}$-Mat(X, Y) has small colimits and $\mathcal{V}$-Mat$(\mathcal{A},\mathcal{B})$ preserves them. Composition with a $\mathcal{V}$-module preserves the small colimits since coequalizers commute with colimits. So $\mathcal{V}$-Mod *is locally small-cocomplete.*

Each $\mathcal{V}$-functor $F : \mathcal{A} \to \mathcal{B}$ determines a $\mathcal{V}$-module $F_* : \mathcal{A} \to \mathcal{B}$ whose underlying matrix is the composite

$$X \xrightarrow[F_*]{} Y \xrightarrow[\mathcal{B}]{} Y,$$

and whose actions ϱ, λ are the composites

$$\mathcal{B}F_*\mathcal{A} \xrightarrow[\mathcal{B}\bar{F}]{} \mathcal{B}\mathcal{B}F_* \xrightarrow[\mu F_*]{} \mathcal{B}F_*, \qquad \mathcal{B}\mathcal{B}F_* \xrightarrow[\mu F_*]{} \mathcal{B}F_*.$$

Modules of the form $F_*: \mathscr{A} \to \mathscr{B}$ have right adjoints $F^*: \mathscr{B} \to \mathscr{A}$. The $\mathscr{W}$-functor F is *fully faithful* if and only if the unit $1_{\mathscr{A}} \to F^*F_*$ is invertible. If the $\mathscr{W}$-functor F is *bijective on objects*, then the counit gives a coequalizer diagram:

$$F_*F^*F_*F^* \rightrightarrows F_*F^* \to 1_{\mathscr{B}}$$

in $\mathscr{W}$-Mod$(\mathscr{B}, \mathscr{B})$; for this is now the Eilenberg–Moore category $\mathscr{W}$-Mod$(\mathscr{A}, \mathscr{B})^{F^*F_*}$. For $\mathscr{W}$-functors $F, G: \mathscr{A} \to \mathscr{B}$, there are natural bijections between 2-cells $F_* \to G_*$ in $\mathscr{W}$-Mod, 2-cells $G^* \to F^*$ in $\mathscr{W}$-Mod, and $\mathscr{W}$-natural transformations $F \to G$.

[We have extended the "hyperdoctrine" Set/$\mathscr{U} \to \mathscr{W}$-Mat of Section 1 to a "hyperdoctrine" $\mathscr{W}$-Cat$\to \mathscr{W}$-Mod.]

As remarked just before Proposition 3, objects A, B of $\mathscr{A}$ over U, V can be regarded as $\mathscr{W}$-functors $A: U \to \mathscr{A}$, $B: V \to \mathscr{A}$. Observe further that $\mathscr{A}(A, B) \cong A^*B_*$. Given a cospan:

$$\mathscr{B} \xrightarrow{G} \mathscr{C} \xleftarrow{F} \mathscr{A}$$

in $\mathscr{W}$-Cat, it is therefore consistent to denote the $\mathscr{W}$-module $G^*F_*: \mathscr{A} \to \mathscr{B}$ by $\mathscr{C}(G, F)$. We shall now see that every $\mathscr{W}$-module has this form.

The mapping cone $\mathrm{Cn}(\Phi)$ of a $\mathscr{W}$-module $\Phi: \mathscr{A} \to \mathscr{B}$ is the $\mathscr{W}$-category defined as follows. Suppose $\mathscr{A}, \mathscr{B}$ are monads on X, Y in $\mathscr{W}$-Mat. Then $\mathrm{Cn}(\Phi)$ is the monad on $Y + X$ made up of the matrix

$$\begin{pmatrix} \mathscr{B} & \Phi \\ 0 & \mathscr{A} \end{pmatrix} : Y + X \to Y + X,$$

with unit

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix},$$

and multiplication

$$\begin{pmatrix} \mu & (\varrho, \lambda) \\ 0 & \mu \end{pmatrix}.$$

In an obvious way we obtain a cospan

$$\mathscr{B} \xrightarrow{J} \mathrm{Cn}(\Phi) \xleftarrow{I} \mathscr{A}$$

in $\mathscr{W}$-Cat, and $\mathrm{Cn}(\Phi)(J, I) = J^*I_* \cong \Phi$.

4. Right liftings and limits

Suppose now that $\mathscr{W}$ satisfies the following further conditions:

C1. Each hom-category $\mathscr{W}(U, V)$ has small limits.

C2. Each pair of arrows $F: U \to W$, $g: V \to W$ admits a right lifting $g \pitchfork f: U \to V$ of f through g:

$$\frac{h \to g \pitchfork f}{gh \to f}$$

Theorem 8. *The bicategories* $\mathcal{V}$-Mat *and* $\mathcal{V}$-Mod *both satisfy conditions* C1 *and* C2.

Proof. Limits in $\mathcal{V}$-Mat(X, Y) can be constructed componentwise so that C1 for $\mathcal{V}$-Mat is easy. For matrices $S: X \to Z$, $T: Y \to Z$, the formula for $T \pitchfork S: X \to Y$ is:

$$(T \pitchfork S)(y, x) = \prod_z T(z, y) \pitchfork S(z, x);$$

with this, C2 is easily checked.

Since $\mathcal{V}$-Mod$(\mathcal{A}, \mathcal{B})$ is monadic over $\mathcal{V}$-Mat(X, Y), limits are carried over; so C1 for $\mathcal{V}$-Mod follows. For modules $\Phi: \mathcal{A} \to \mathcal{C}$, $\Psi: \mathcal{B} \to \mathcal{C}$, the module $\Psi \pitchfork \Phi: \mathcal{A} \to \mathcal{B}$ is made up of the equalizer in $\mathcal{V}$-Mat of the two arrows:

$$\begin{array}{ccc} \Psi \pitchfork \Phi & \xrightarrow{\lambda \pitchfork 1} & \mathcal{C}\Psi \pitchfork \Phi \\ & \searrow^{\varrho} \qquad \nearrow_{1 \pitchfork \lambda} & \\ & \mathcal{C}\Psi \pitchfork \mathcal{C}\Phi & \end{array}$$

the ϱ induced by the ϱ of Φ, and the λ induced by the ϱ of Ψ. Condition C2 for $\mathcal{V}$-Mod is easily checked. □

For each $\mathcal{V}$-category $\mathcal{B}$ based on the category Set of small sets, there is a $\mathcal{V}$-cateogry $\mathcal{P}\mathcal{B}$ based on SET, defined as follows:

$$(\mathcal{P}\mathcal{B})_U = \mathcal{V}\text{-Mod}(U, \mathcal{B}), \qquad (\mathcal{P}\mathcal{B})(\Psi, \Phi) = \Psi \pitchfork \Phi.$$

There is a pseudo-natural equivalence:

$$\mathcal{V}\text{-Mod}(\mathcal{A}, \mathcal{B}) \simeq \mathcal{V}\text{-CAT}(\mathcal{A}, \mathcal{P}\mathcal{B}).$$

Precisely the same arguments used in proving Proposition 1 now yield:

Proposition 9. *The homomorphism* $\mathcal{V}$-Cat $\to$ $\mathcal{V}$-Mod *satisfies all the properties listed for the homomorphism* Set$/\mathcal{V} \to \mathcal{V}$-Mat *in Proposition* 1. □

Theorem 10. *The 2-category* $\mathcal{V}$-Cat *has all small limits.*

Proof. Suppose $J: \mathcal{J} \to$ Cat, $D: \mathcal{J} \to \mathcal{V}$-Cat are functors from a small category $\mathcal{J}$. Write D_U for the composite of D with $\mathcal{V}$-Cat$(U, -)$: $\mathcal{V}$-Cat$\to$Cat. Define a $\mathcal{V}$-

category $\mathscr{S}$ as follows. An object of $\mathscr{S}$ over U is a natural transformation $A: J \to D_U$. For objects A, B of $\mathscr{S}$ over U, V, take $\mathscr{S}(B, A)$ to be the limit in $\mathscr{W}(U, V)$ of the diagram:

$$\begin{array}{ccc} (DC)(B_C j, A_C j) & & (DC')(B_{C'} j', A_{C'} j\ ) \\ {\scriptstyle (DC')(1, A_{C'}\xi)\cdot D_U n} \searrow & & \swarrow {\scriptstyle (DC')(B_{C'}\xi, 1)} \\ & (DC')(B_{C'}(Jn)j, A_{C'} j') & \end{array}$$

as n, ξ run over arrows $n: C \to C'$, $\xi: (Jn)j \to j'$ in $\mathscr{C}$, JC', respectively. One may verify the isomorphism

$$(\mathscr{W}\text{-Cat})(\mathscr{X}, \mathscr{S}) \cong [\mathscr{C}, \mathrm{Cat}](J, \mathscr{W}\text{-Cat}(\mathscr{X}, D)). \qquad \square$$

5. Fibrations as enriched categories

Let $\mathbf{C}$ denote a small category whose set of objects is $\mathscr{V}$. Rather than the 2-category of fibrations over $\mathbf{C}$, we prefer to deal with the equivalent 2-category

$$\mathscr{H}(\mathbf{C}) = \mathrm{Hom}(\mathbf{C}^{\mathrm{op}}, \mathrm{Cat})$$

of homomorphisms from $\mathbf{C}^{\mathrm{op}}$ to Cat and strong (=pseudo-natural) transformations between them. We identify the category $\hat{\mathbf{C}} = [\mathbf{C}^{\mathrm{op}}, \mathrm{Set}]$ of presheaves on $\mathbf{C}$ with a full sub-2-category of $\mathscr{H}(\mathbf{C})$ consisting of discrete objects. We also regard $\mathbf{C}$ as a full sub-2-category of $\mathscr{H}(\mathbf{C})$ consisting of representable objects.

Recall the construction of the bicategory Spn $\mathscr{A}$ from a category $\mathscr{A}$ with pullbacks (Bénabou [1, p. 22]). Our convention is to draw a span S from U to V as

$$V \longleftarrow S \longrightarrow U,$$

and to identify an arrow $f: U \to V$ in $\mathscr{A}$ with the span

$$V \overset{f}{\longleftarrow} U \overset{1}{\longrightarrow} U.$$

It is a straightforward calculation to verify the following assertion (the case $\mathscr{A} = \mathrm{Set}$ suffices):

An arrow S in Spn $\mathscr{A}$ *has a right adjoint if and only if it is isomorphic to an arrow in $\mathscr{A}$.*

Let $\mathscr{W}(\mathbf{C})$ denote the full subbicategory of Spn $\hat{\mathbf{C}}$ determined by the objects which are actually in $\mathbf{C}$. Arrows in $\mathscr{W}(\mathbf{C})$ are spans in $\hat{\mathbf{C}}$ between objects of $\mathbf{C}$.

An arrow in $\mathscr{W}(\mathbf{C})$ has a right adjoint if and only if it is isomorphic to an arrow in $\mathbf{C}$. (This follows from the above assertion about adjunctions in Spn $\mathscr{A}$ and the Yoneda Lemma.)

The properties required of $\mathscr{W}$ in Section 1 and properties C1, C2 of Section 4 are satisfied by $\mathscr{W}(\mathbf{C})$ since $\hat{\mathbf{C}}$ is a Grothendieck topos.

Our purpose now is to study the relationship between $\mathscr{M}(\mathbf{C})$ and $\mathscr{W}(\mathbf{C})$-Cat. This study begins with the 2-functor

$$L:\mathscr{M}(\mathbf{C})\to\mathscr{W}(\mathbf{C})\text{-Cat}$$

defined below.

Each object T of $\mathscr{M}(\mathbf{C})$ determines a $\mathscr{W}(\mathbf{C})$-category LT defined as follows. An object of LT over U is an object of TU which we can also view as an arrow $U\to T$ in $\mathscr{M}(\mathbf{C})$ (using the bicategorical Yoneda lemma). For objects x, y of TU, TV, the arrow $(LT)(x,y):V\to U$ in $\mathscr{W}(\mathbf{C})$ is the span from V to U obtained as the comma object of $x:U\to T$, $y:V\to T$ in $\mathscr{M}(\mathbf{C})$:

$$\begin{array}{ccc} (LT)(x,y) & \xrightarrow{\ d_1\ } & V \\ {\scriptstyle d_0}\downarrow & \overset{\lambda}{\Rightarrow} & \downarrow{\scriptstyle y} \\ U & \xrightarrow[\ x\ ]{} & T \end{array}$$

Since U, V have values in Set, so does $(LT)(x,y)$. More explicitly,

$$(LT)(x,y)S=\{(u,\theta,v)\mid u:S\to U,\ v:S\to V \text{ in } \mathbf{C} \text{ and}$$
$$\theta:(Tu)x\to(Tv)y \text{ in } TS\}.$$

Composition for LT is given by:

$$((LT)(x,y)\circ(LT)(y,z))S\to(LT)(x,z)S,$$
$$((u,\theta,v),(v,\phi,w))\mapsto(u,\phi\theta,w).$$

For each arrow $\sigma:T\to T'$ in $\mathscr{M}(\mathbf{C})$, there is a $\mathscr{W}(\mathbf{C})$-functor $L\sigma:LT\to LT'$. The object x of LT over U is taken to $(L\sigma)x=\sigma_U x$, and the function

$$(L\sigma)_{xy}S:(LT)(x,y)S\to(LT')(\sigma_U x,\sigma_V y)S$$

takes (u,θ,v) to (u,θ',v), where θ' is the composite

$$(T'u)(\sigma_U x)\cong\sigma_S(Tu)x\xrightarrow[\sigma_S(\theta)]{}\sigma_S(Tv)y\cong(T'v)(\sigma_V y).$$

Theorem 11. *The 2-functor* $L:\mathscr{M}(\mathbf{C})\to\mathscr{W}(\mathbf{C})$-Cat *has a right adjoint with fully faithful unit.*

Proof. Since $\mathbf{C}$ is a small full dense sub-2-category of $\mathscr{M}(\mathbf{C})$ and $\mathscr{W}(\mathbf{C})$-Cat is small cocomplete (Theorem 7), a right adjoint Γ for L must have the form:

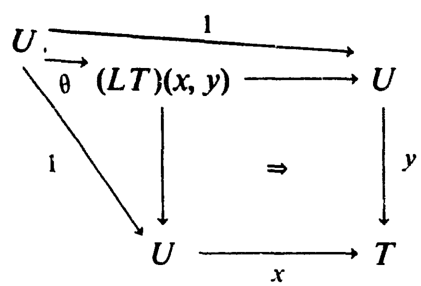
$$\Gamma\mathscr{A}=\mathscr{W}(\mathbf{C})\text{-Cat}(L-,\mathscr{A}):\mathbf{C}^{\mathrm{op}}\to\mathrm{Cat}.$$

The unit $\eta:1\to\Gamma L$ has component at T given by the composite:

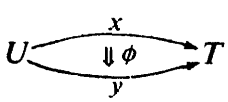
$$T-\simeq\mathscr{M}(\mathbf{C})(-,T)\xrightarrow{\ L\ }\mathscr{W}(\mathbf{C})\text{-Cat}(L-,LT)=\Gamma(LT)-.$$

There is a $\mathscr{V}$-functor $i_U: U \to LU$ for each object U of $\mathbf{C}$ which takes the one object of U to 1_U as an object of LU over U. (The objects of LU over V are arrows $V \to U$ in $\mathbf{C}$.)

To see that $\eta_T: T \to \Gamma LT$ is fully faithful, take $x, y: U \to T$ in $\mathscr{H}(\mathbf{C})$ and $\theta: Lx \Rightarrow Ly$ in $\mathscr{V}(\mathbf{C})$-Cat. This gives $\theta i_U: x = (Lx)i_U \to (Ly)i_U = y$ in $(LT)_U$, which means a map of spans $1_U \to (LT)(x, y)$ from U to U:



Thus we obtain a unique 2-cell



in $\mathscr{H}(\mathbf{C})$ with $L\phi = \theta$. This completes the proof that η_T is fully faithful.

The $\mathscr{V}(\mathbf{C})$-functor $\varepsilon_{\mathscr{A}}: L\Gamma\mathscr{A} \to \mathscr{A}$ takes an object $A: LU \to \mathscr{A}$ over U to the object of $\mathscr{A}$ over U corresponding to $Ai_U: U \to \mathscr{A}$. Given $A: LU \to \mathscr{A}$, $B: LV \to \mathscr{B}$ in $\mathscr{V}(\mathbf{C})$-Cat, we must describe an arrow of spans

$$(L\Gamma\mathscr{A})(A, B) \to \mathscr{A}(Ai_U, Bi_V)$$

from V to U in $\hat{\mathbf{C}}$. Elements of $(L\Gamma\mathscr{A})(A, B)S$ are triples (u, θ, v) where u, v make S into a span in $\mathbf{C}$ from V to U and $\theta: A \cdot Lu \Rightarrow B \cdot Lv$ is a $\mathscr{V}(\mathbf{C})$-natural transformation. Composing with $i_S: S \to LS$, we obtain a 2-cell $(A \cdot i_U)_* u \Rightarrow (B \cdot i_V)_* v$ in $\mathscr{V}(\mathbf{C})$-Mod. This gives a 2-cell $uv^* \Rightarrow (Ai_U)^*(Bi_V)_*$ in $\mathscr{V}(\mathbf{C})$-Mod between arrows from V to U. But $\mathscr{V}(\mathbf{C})\text{-Mod}(V, U) = \mathscr{V}(\mathbf{C})(V, U)$. So we have an element of $\mathscr{A}(Ai_U, Bi_V)S \cong (Ai_U)^*(Bi_V)_*S$.

The adjunction identities can be routinely checked. □

Theorem 12. *The 2-functor* $L: \mathscr{H}(\mathbf{C}) \to \mathscr{V}(\mathbf{C})$-Cat *preserves small limits.*

Proof. Since the construction of L involves comma objects which are themselves limits in $\mathscr{H}(\mathbf{C})$, the verification is routine. □

6. Cofibrations and cauchy completeness

For any small bicategory $\mathscr{H}$, *fibrations in* $\mathrm{Hom}(\mathscr{H}^{\mathrm{op}}, \mathrm{Cat})$ were extensively studied in [11]. A bicategory $\mathrm{DFib}(\mathrm{Hom}(\mathscr{H}^{\mathrm{op}}, \mathrm{Cat}))$ was constructed having the same objects as $\mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{Cat})$ and having the bidiscrete fibrations as arrows. To each homomorphism $T: \mathscr{K}^{\mathrm{op}} \to \mathrm{Cat}$ was associated its *cooperative homomorphism* $\# T: \mathscr{K}^{\mathrm{co}} \to \mathrm{Cat}$ which provided the following *representation of bidiscrete fibrations:*

$$\mathrm{DFib}(\mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{Cat}))(S, T) \simeq \mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{CAT})(S, [(\# T)^{\mathrm{op}}, \mathrm{Set}]).$$

A fibration in $\mathscr{K}$ is a span in $\mathscr{K}$ which is taken to a fibration by $\mathscr{K} \to \mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{Cat})$. This agrees with the definition in [10] where the fibration property is expressed in terms of finite bilimits in $\mathscr{K}$. A finitely bicomplete and finitely bicocomplete bicategory $\mathscr{K}$ was called *fibrational* when bipullback along a leg of a fibration preserved the bicolimit involved in the definition of fibrational composition. Under these conditions one obtained a bicategory $\mathrm{DFib}(\mathscr{K})$ with the same objects as $\mathscr{K}$ and with bidiscrete fibrations in $\mathscr{K}$ as arrows.

By a change of universe, the construction of $\mathrm{DFib}(\mathscr{K})$ can be made even when $\mathscr{K}$ is not small and agrees with that of the first paragraph of this section when applied to $\mathrm{Hom}(\mathscr{K}^{\mathrm{op}}, \mathrm{Cat})$.

Fibrations in $\mathscr{K}^{\mathrm{op}}$ are called *cofibrations in* $\mathscr{K}$, and bidiscrete fibrations in $\mathscr{K}^{\mathrm{op}}$ will be called *modules in* $\mathscr{K}$. When $\mathscr{K}^{\mathrm{op}}$ is fibrational, we obtain a bicategory $\mathrm{DFib}(\mathscr{K}^{\mathrm{op}})$; set

$$\mathrm{Mod}(\mathscr{K}) = \mathrm{DFib}(\mathscr{K}^{\mathrm{op}})^{\mathrm{co}}.$$

If both $\mathscr{K}$ and $\mathscr{K}^{\mathrm{op}}$ and fibrational, there is a homomorphism

$$\mathrm{Mod}(\mathscr{K}) \to \mathrm{DFib}(\mathscr{K})$$

which is the identity on objects and which takes each module to the bicomma object of its underlying cospan. The dual construction gives a left biadjoint for this homomorphism.

Theorem 13. *For any small category* $\mathbf{C}$, *the bicategories* $\mathscr{K}(\mathbf{C})$, $\mathscr{K}(\mathbf{C})^{\mathrm{op}}$ *are both fibrational and the homomorphism of the last paragraph provides a biequivalence:*

$$\mathrm{Mod}(\mathscr{K}(\mathbf{C})) \sim \mathrm{DFib}(\mathscr{K}(\mathbf{C})).$$

Proof. It was proved in [10] that Cat and $\mathrm{Cat}^{\mathrm{op}}$ are both fibrational. Every module in Cat is the cocomma object of its comma object. This gives the result for "constant categories" ($\mathbf{C} = 1$). The "variable" case is then straightforward after [11; 3.8]. □

Theorem 14. *Suppose* $\mathscr{V}$ *is a locally small-cocomplete bicategory with a small set of objects that satisfies* C1, C2 *of Section* 4. *Then* $(\mathscr{V}\text{-Cat})^{\mathrm{op}}$ *is a fibrational bicategory and there is a biequivalence*

$$\mathscr{V}\text{-Mod} \sim \mathrm{Mod}(\mathscr{V}\text{-Cat})$$

which is the identity on objects and takes each $\mathscr{V}$*-module to its mapping cone.*

Proof. The case where $\mathscr{W}$ has one object was dealt with in [10; §6]. The generalization here provides no difficulties. □

A module from A to B in a bicategory $\mathscr{K}$ is called *cauchy* when it has a right adjoint in Mod($\mathscr{K}$). A module from A to B in $\mathscr{K}$ is called *convergent* when there exists an arrow $f: A \to B$ in $\mathscr{K}$ such that the module is equivalent to the bicocomma object of the span

$$B \xleftarrow{f} A \xrightarrow{1_A} A.$$

Every convergent module is cauchy. Call an object B of $\mathscr{K}$ *cauchy-complete* when every cauchy module into B is convergent. Write $\mathscr{K}_{\mathrm{cc}}$ for the full subbicategory of $\mathscr{K}$ consisting of the cauchy-complete objects.

Corollary 15. *The 2-functor* $L: \mathscr{K}(\mathbf{C}) \to \mathscr{W}(\mathbf{C})$-Cat *induces a homomorphism of bicategories*

$$L: \mathrm{Mod}\, \mathscr{K}(\mathbf{C}) \to \mathscr{W}(\mathbf{C})\text{-Mod}.$$

Proof. Since Mod($\mathscr{K}$) is constructed from $\mathscr{K}$ using finite bilimits and finite bicolimits, the result follows from Theorems 11, 12, 14. □

Proposition 16. (a) *An object T of $\mathscr{K}(\mathbf{C})$ is cauchy-complete if and only if, for all objects W of* $\mathbf{C}$, *idempotents split in the category TW.*

(b) *An object $\mathscr{A}$ of $\mathscr{W}$-*Cat *is cauchy-complete if and only if, for all objects W of $\mathscr{W}$, each cauchy $\mathscr{W}$-module $W \to \mathscr{A}$ is convergent.*

Proof. Part (b) follows from the fact that the objects W of $\mathscr{W}$-Cat can be used to detect convergence of modules; as a special case, an object of Cat is cauchy-complete if and only if each module from 1 into it is convergent. It can be calculated from this (as is well known) that cauchy-complete categories are those in which idempotents split.

To prove part (a), take $T \in \mathscr{K}(\mathbf{C})$. Suppose idempotents split in each TW. For each object W of $\mathbf{C}$, the evaluation homomorphism $\mathrm{ev}_W: \mathscr{K}(\mathbf{C}) \to \mathrm{Cat}$ preserves finite limits and colimits, and so an arrow $E: X \to T$ with a right adjoint E^* in Mod($\mathscr{K}(\mathbf{C})$) gives an arrow $E_W: XW \to TW$ with a right adjoint in Mod(Cat). Since TW is a cauchy-complete category, there exists a functor $f_W: XW \to TW$ such that E_W, E^*_W are isomorphic to the discrete fibrations associated with the comma categories $TW/f_W, f_W/TW$, respectively. Since E, E^* are homomorphisms, it follows that the functors f_W are the components of a strong transformation $f: X \to T$. Clearly E converges to f. So T is cauchy-complete.

Conversely, suppose T is cauchy-complete in $\mathscr{K}(\mathbf{C})$. An idempotent in TW amounts to an idempotent in $\mathscr{K}(\mathbf{C})(W, T)$. This gives an idempotent between convergent modules whose splitting gives a cauchy module $W \to T$. Since T is cauchy-complete, this splitting converges giving a splitting of the idempotent in TW. □

7. The main biequivalence

For each object T of $\mathscr{M}(\mathbf{C})$, there is a homomorphism of bicategories $\mathscr{P}T:\mathbf{C}^{\mathrm{op}}\to\mathrm{CAT}$ whose value at W is given by

$$(\mathscr{P}T)W=\mathrm{Mod}\,\mathscr{M}(\mathbf{C})(W,T)\simeq[(W\# T)^{\mathrm{op}},\mathrm{Set}].$$

This determines a homomorphism

$$\mathscr{P}:\mathscr{M}(\mathbf{C})^{\mathrm{coop}}\to\mathrm{Hom}(\mathbf{C}^{\mathrm{op}},\mathrm{CAT})$$

which is part of a Yoneda structure [11; §6]. Recall also the definition of $\mathscr{P}$ for enriched categories given earlier (Section 4). For each $T\in\mathscr{M}(\mathbf{C})$, there is a comparison $\mathscr{M}(\mathbf{C})$-functor

$$L\mathscr{P}T\to\mathscr{P}LT$$

determined using the fact that both $\mathscr{P}$'s represent modules and using Corollary 15.

For the next result it is helpful to use the explicit description of $\#T:\mathbf{C}\to\mathrm{Cat}$ for $T\in\mathscr{M}(\mathbf{C})$. The value of $\#T$ at $W\in\mathbf{C}$ is the category $W\# T$ whose objects are pairs (f,x) where $f:U\to W$ is an arrow in $\mathbf{C}$ and $x\in TU$, and whose arrows $(h,\xi):(f,x)\to(f',x')$ consist of $h:U\to U'$ in $\mathbf{C}$, $\xi:x\to(Th)x'$ in TU with $f=f'h$.

Proposition 17. *The 2-functor*

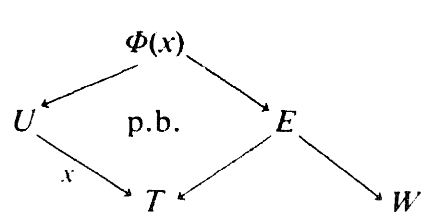
$$L:\mathrm{Hom}(\mathbf{C}^{\mathrm{op}},\mathrm{CAT})\to\mathscr{M}(\mathbf{C})\text{-CAT}$$

is a logical morphism of Yoneda structures; in other words, the comparison arrow is an equivalence

$$L\mathscr{P}T\simeq\mathscr{P}LT.$$

It follows that L takes cauchy-complete objects of $\mathscr{M}(\mathbf{C})$ into cauchy-complete $\mathscr{M}(\mathbf{C})$-categories.

Proof. The comparison arrow $(L\mathscr{P}T)_W\to(\mathscr{P}LT)_W$ takes a bidiscrete fibration E from W to T to the $\mathscr{M}(\mathbf{C})$-module Φ from W to LT given by $\Phi(x)=x^*E$:



On the other hand, a $\mathscr{W}(\mathbf{C})$-module Φ from W to LT determines a functor $E:(W \# T)^{\mathrm{op}} \to \mathrm{Set}$ whose value at (f, x) is given by

$$E(f, x) = \mathscr{W}(\mathbf{C})(W, U)(f^*, \Phi(x)).$$

Clearly the bidiscrete fibration corresponding to this E (under the representation theorem) is taken to an isomorph of Φ under the comparison arrow. The remaining details are easily checked. □

Proposition 18. *The right adjoint Γ of L preserves cauchy completeness.*

Proof. Let $\mathscr{A}$ be a cauchy-complete $\mathscr{W}(\mathbf{C})$-category. Then

$$(\Gamma\mathscr{A})U = \mathscr{W}(\mathbf{C})\text{-Cat}(LU, \mathscr{A}) = \mathscr{W}(\mathbf{C})\text{-Cat}(U, \mathscr{A})$$

(since LU is the cauchy-completion of U), which is equivalent to the full subcategory of $\mathscr{W}(\mathbf{C})\text{-Mod}(U, \mathscr{A})$ consisting of the cauchy modules. Now $\mathscr{W}(\mathbf{C})$-Mod$(U, \mathscr{A})$ is small cocomplete, so certainly idempotents split therein. Suppose $\Phi : U \to \mathscr{A}$ is cauchy and $\varrho : \Phi \to \Phi$ is an idempotent. Then we have a corresponding idempotent $\varrho^* : \Phi^* \to \Phi^*$ on the right adjoint of Φ. A splitting for ϱ^* gives a right adjoint for a splitting of ϱ. □

Theorem 19. *The 2-functor $L : \mathscr{H}(\mathbf{C}) \to \mathscr{W}(\mathbf{C})$-Cat restricts to a biequivalence*

$$\mathscr{H}(\mathbf{C})_{\mathrm{cc}} \simeq (\mathscr{W}(\mathbf{C})\text{-Cat})_{\mathrm{cc}}.$$

Proof. The unit $\eta : 1 \to \Gamma L$ is fully faithful (Theorem 11). If T is cauchy-complete in $\mathscr{H}(\mathbf{C})$ then LT is cauchy complete. Since LU is the cauchy-completion of U, we have:

$$\mathscr{W}(\mathbf{C})\text{-Cat}(LU, LT) \simeq \mathscr{W}(\mathbf{C})\text{-Cat}(U, LT)$$

which has the same objects as TU. It follows that η_T is surjective on objects up to isomorphism.

Suppose $\mathscr{A}$ is a cauchy-complete $\mathscr{W}(\mathbf{C})$-category. Objects of $L\Gamma\mathscr{A}$ are $\mathscr{W}(\mathbf{C})$-functors $U \to \mathscr{A}$, which amounts to objects of $\mathscr{A}$. Take two objects A, B of $\mathscr{A}$ over U, V, respectively. To give a 2-cell

$$\begin{array}{ccc} S & \xrightarrow{\ v\ } & V \\ {\scriptstyle u}\big\downarrow & \overset{\theta}{\Rightarrow} & \big\downarrow{\scriptstyle B} \\ U & \xrightarrow[A]{} & \Gamma\mathscr{A} \end{array}$$

in $\mathscr{S}(\mathbf{C})$ is precisely to give a 2-cell

$$\begin{array}{ccc} S & \xrightarrow{v} & V \\ {\scriptstyle u}\downarrow & \Rightarrow & \downarrow{\scriptstyle B_*} \\ U & \xrightarrow[A_*]{} & \mathscr{A} \end{array}$$

in $\mathscr{S}(\mathbf{C})$-Mod. But a 2-cell $A_*u \Rightarrow B_*v$ amounts to a 2-cell $uv^* \Rightarrow A^*B_*$ in $\mathscr{S}(\mathbf{C})$-Mod. This is the same as a map of spans $S \to \mathscr{A}(A, B)$. So $(L\Gamma\mathscr{A})(A, B) \cong \mathscr{A}(A, B)$. Thus $\varepsilon_{\mathscr{A}}$ is an equivalence. □

Let Rel(**C**) denote the bicategory whose objects are the objects of **C**, whose arrows are relations in $\hat{\mathbf{C}}$ between the representables, whose 2-cells are inclusions, and whose composition is the usual composition of relations. There is a homomorphism of bicategories

$$\mathscr{S}(\mathbf{C}) \to \mathrm{Rel}(\mathbf{C})$$

which is the identity on objects and is given on hom-categories by the reflection of spans into relations.

Corollary 20. *The 2-functor L induces a biequivalence of 2-categories*

$$[\mathbf{C}^{\mathrm{op}}, \mathrm{Poset}] \sim (\mathrm{Rel}(\mathbf{C})\text{-}\mathrm{Cat})_{\mathrm{cc}}$$

where Poset *denotes the 2-category of small ordered sets.* □

The result of Walters [14] characterizing presheaves on **C** as *symmetric* cauchy-complete Rel(**C**)-categories is obtained on restriction of the biequivalence of Corollary 20.

References

[1] J. Bénabou, Introduction to bicategories, Lecture Notes in Math. 47 (Springer, Berlin, 1967) 1-77.

[2] J. Bénabou, Les distributeurs, Univ. Catholique de Louvain, Séminaires de Math. Pure, Rapport No. 33 (1973).

[3] R. Betti and A. Carboni, Cauchy completion and the associated sheaf, Cahiers de Topologie et Géom. Diff. 23 (1982) 243-256.

[4] R. Betti and A. Carboni, Notion of topology for bicategories, Cahiers de Topologie et Géom. Diff. (1983) to appear.

[5] G.M. Kelly and R.H. Street, Review of the elements of 2-categories, Lecture Notes in Math. 420 (Springer, Berlin, 1974) 75-103.

[6] F.W. Lawvere, Closed categories and biclosed bicategories, Lectures at Math. Inst. Aarhus Universitet (Fall, 1971).

[7] F.E.J. Linton, Coequalizers in categories of algebras, Lecture Notes in Math. 80 (Springer, Berlin, 1969) 75-90.

[8] S. MacLane, Categories for the Working Mathematician (Springer, New York-Heidelberg-Berlin, 1971).
[9] R.H. Street, The formal theory of monads, J. Pure Appl. Algebra 2 (1972) 149-168.
[10] R.H. Street, Fibrations in bicategories, Cahiers de Topologie et Géom. Diff. 21 (1980) 111-160.
[11] R.H. Street, Conspectus of variable categories, J. Pure Appl. Algebra 21 (1981) 307-338.
[12] R.H. Street, Enriched categories and cohomology, Quaestiones Math. (1983), to appear.
[13] R.H. Street and R.F.C. Walters, Yoneda structures on 2-categories, J. Algebra 50 (1978) 350-379.
[14] R.F.C. Walters, Sheaves on sites as Cauchy-complete categories, J. Pure Appl. Algebra 24 (1982) 95-102.
[15] H. Wolff, $\mathscr{V}$-cat and $\mathscr{V}$-graph, J. Pure Appl. Algebra 4 (1974) 123-135.